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Duality, Multi-Monopole Dynamics & Quantum Threshold Bound States[1]

Piljin Yi [2]

Physics Department, Columbia University, New York, NY 10027

Dynamics of supersymmetric monopoles are studied in the low energy approximation. A conjecture for the exact moduli space metric is given for all collections of fundamental monopoles of distinct type, and various partial confirmations of the conjecture are outlined. Upon the quantization of the resulting multi-monopole dynamics in the context of $N = 4$ supersymmetric Yang-Mills-Higgs theories, one recovers the missing magnetic states that are dual to some of the massive vector mesons. A generalization to monopoles with nonabelian charges is also discussed.

In this talk, we are primarily interested in two aspects of supersymmetric monopoles. The first is the classical low energy interactions between them, which can be encoded in the geometry of the so-called moduli space [1]. This moduli space approach turns out to be not only a powerful tool in probing the interactions, but also very convenient in quantization of the low energy dynamics, which brings us to our second goal: testing the hypothesized self-duality of all $N = 4$ supersymmetric Yang-Mills-Higgs models.

The duality hypothesis [2, 3] for this class of theories says that the elementary electric part of spectrum and the solitonic magnetic part of spectrum are mirror images of each other. That is, unless the gauge group contains a factor of $SO(2N + 1)$ or $Sp(2N)$, in which case the electric spectrum of $SO(2N + 1)$ theory is the mirror image of the magnetic spectrum of $Sp(2N)$ theory, and vice versa. However, as soon as one looks into the solitons of these theories, a puzzle arises. For simplicity, suppose the gauge group is simple. A generic adjoint Higgs will break the gauge group

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[2] e-mail: piljin@phys.columbia.edu

to its Cartan torus $U(1)^k$ where k is the rank of G, upon which topological solitons with magnetic $U(1)$ charges can be created. The topological charges take values in the second homotopy group of the vacuum manifold $\pi_2(G/U(1)^k) = \pi_1(U(1)^k) = \mathbf{Z}^k$, and thus there are at most k independent topological charges, which in turn suggests that there are only k species of magnetic monopoles. This can be also seen from the remarkable result by E. Weinberg [4], who counted the number of zero modes, or the number of ways to deform a given configuration and showed that solutions with higher topological charges should always be considered as a collection of more than one monopoles.

On the other side of the coin, the electrically charged vector mesons are simply gauge particles that become massive through the Higgs effect. Since the unbroken group is $U(1)^k$, only k photons may stay massless and the remaining ($\dim\, G - k$) gauge particles must acquire mass. But the number of these complex massive vector mesons, $(\dim\, G - k)/2$, is equal to the number of fundamental monopoles k only for the gauge group $G = SU(2)$. For all other simple gauge groups, $(\dim\, G - k)/2 - k$ number of magnetic states appear to be missing. And this is the problem we want to address in this talk.

Of course, the above comparison does not really make sense since the counting of the solitons is completely classical while duality is an intrinsically quantum statement. One must look beyond the classical states of the solitons, and consider their quantum counterpart as well in order to perform a meaningful test of duality [5, 6]. Thus the question is whether the quantum mechanics of these fundamental monopoles are such that the missing magnetic states are all recovered as their quantum bound states.

One point that deserves an emphasis here is that the bound states in question have to be all threshold bound states, i.e, without any binding energy. This can be easily seen from the usual BPS mass formula [7],

$$\begin{aligned} E &\geq \left(P^2 + Q^2\right)^{1/2}, \\ P &= \oint (\mathrm{tr}\, \Phi B), \\ Q &= \oint (\mathrm{tr}\, \Phi E), \end{aligned} \tag{1}$$

where Φ is the adjoint Higgs field and B and E are magnetic and electric field strengths respectively. For vector mesons of purely electric charges ($P = 0$) or for monopoles of purely magnetic charges

($Q = 0$), the mass formula is obviously additive, so the mass of a composite state is given by sum of the masses of individual components.

The absence of the binding energy complicates the problem further, for it means one must understand the monopole interaction very precisely in order to uncover the bound state spectrum. In other words, we must know the exact geometry of the multi-monopole moduli spaces.

The exact moduli space geometry for a pair of identical $SU(2)$ monopoles has been known for quite some time [8]. Atiyah and Hitchin translated the symmetries of BPS equations into those of the moduli space, and thereby succeeded in isolating the exact moduli space metric. But as we mentioned above, our problem arises with gauge groups larger than $SU(2)$, and furthermore requires understanding of interaction among many monopoles, not just a pair. At first, the prospect of finding the exact moduli spaces for any number of monopoles that arise from arbitrary gauge groups, appears quite hopeless. However, it turns out that there is a key difference from the problem Atiyah and Hitchin confronted: we are primarily interested in collections of distinct monopoles.

For instance, with $G = SU(3)$ there exists two distinct fundamental monopoles, which generate $\pi_2(SU(3)/U(1)^2) = \mathbf{Z}^2$. The complex vector mesons on the other hand come in three varieties, with the two least massive in one-to-one correspondence with these fundamental monopoles. The third, most massive vector mesons has quantum numbers that are sums of those of the other two, so the missing magnetic counterpart has to be a bound state of two distinct monopoles. In fact for $G = SU(k+1)$, all the expected bound states are composed of such collections of n ($\leq k$) distinct fundamental monopoles.

Exactly how the two cases are different in practical terms? Since one can always excite electric charges on the monopole, let us consider a pair of dyons being scattered off each other, with the two topological charges either identical or distinct. At large mutual separations, each dyon behaves as a point-like particle that carries $U(1)^k$ charges and interacts simply by exchanging the uncharged massless photons. The corresponding asymptotic approximation can be easily found in the low energy limit, where the problem becomes that of point charges interacting through induced electromagnetic fields of each other. Other than a sign, the asymptotic form of interaction is more or less the same for the identical pair and the distinct pair.

As their separation becomes smaller and smaller, however, a qualitative difference arises. If

the topological charges of the pair were identical, there will be effectively only one unbroken $U(1)$ conservation law, so there is no reason why individual electric charges should be conserved. In fact, the interaction between the monopole cores allows electric charge to hop from one dyon to the other [8]. This is in a stark contrast with the case of distinct topological charges, since in the latter case each dyon is equipped with its own unbroken $U(1)$ conservation law and the individual electric charges are preserved. More generally, n distinct monopoles come with n independent $U(1)$ symmetries [9, 10].

The asymptotic approximation to multi-monopole dynamics is in general unreliable because it does not take into account what happens when the soliton cores begin to overlap with each other [11]. For any number of distinct monopoles, however, the extra $U(1)$ symmetries ensure that at least one aspect of such a short distance interaction, the electric charge transfer, is absent. This raises the possibility that the asymptotic form of the interaction is in fact exact.

Let us first define the notations we use [10]. Write the Lie algebra of a given simple group G in terms of k Cartan generators $H_i, i = 1, \ldots, k$ normalized as

$$\mathrm{tr}\, H_i H_j = \delta_{ij}, \tag{2}$$

and $(\dim G - k)$ number of ladder operators $E_{\boldsymbol{\alpha}}$, satisfying

$$[H_i, E_{\boldsymbol{\alpha}}] = \alpha_i E_{\boldsymbol{\alpha}}, \qquad [E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}] = \alpha_i H_i. \tag{3}$$

The k-dimensional vectors $\boldsymbol{\alpha}$ are the roots of the Lie algebra. A maximal symmetry breaking to $U(1)^k$ is achieved by allowing an adjoint Higgs to take an expectation value so that only k number of gauge particles remain massless. In the unitary gauge, where the Higgs expectation is "diagonalized"

$$\langle \Phi \rangle = h_i H_i, \tag{4}$$

the gauge bosons associated with the ladder operator $E_{\boldsymbol{\alpha}}$ acquire a mass $|e\alpha_i h_i|$, so the maximal breaking is achieved when $\alpha_i h_i$ is nonzero for all roots $\boldsymbol{\alpha}$. Such an h picks a preferred direction on the root space and a preferred definition of positivity on the root lattice. This in turn leads to a unique set of (positive) simple roots $\{\boldsymbol{\beta}_a, a = 1, \ldots, k\}$ satisfying

$$h_i \beta_{ai} > 0, \tag{5}$$

and

$$\boldsymbol{\alpha} = \sum_a n_a \boldsymbol{\beta}_a \tag{6}$$

with the integers $\{n_a\}$ either all nonnegative or all nonpositive.

On the solitonic side, fundamental monopoles that each carries a unit topological charge are given by classical solutions of magnetic charge [4]

$$\mathbf{g} = \frac{4\pi}{e}\boldsymbol{\beta}_a^* \equiv \frac{4\pi}{e}\frac{\boldsymbol{\beta}_a}{\boldsymbol{\beta}_a^2}. \tag{7}$$

Such a solution can be found by embedding the usual $SU(2)$ BPS solution to the G gauge theory along the $SU(2)$ subgroup spanned by $\{\beta_{ai}H_i, E_{\boldsymbol{\beta}_a}, E_{-\boldsymbol{\beta}_a}\}$ up to a normalization. More generally, a monopole solution carrying a magnetic charge [12]

$$\mathbf{g} = \frac{4\pi}{e}\boldsymbol{\alpha}^* \equiv \frac{4\pi}{e}\frac{\boldsymbol{\alpha}}{\boldsymbol{\alpha}^2}. \tag{8}$$

is found by a similar embedding for any root $\boldsymbol{\alpha}$. One can write any such composite magnetic charges as an integer sum of the fundamental charges [3, 4],

$$\boldsymbol{\alpha}^* = \sum_a \tilde{n}_a \boldsymbol{\beta}_a^* \tag{9}$$

again with the integers $\{\tilde{n}_a\}$ either all nonnegative or all nonpositive. One way to set apart the fundamental monopole from the rest is to consider the number of bosonic collective coordinates:

$$4\,|\sum_a \tilde{n}_a|. \tag{10}$$

The number 4 represents three translation modes as well as a single $U(1)$ angle, so one can separate such a configuration into $|\sum_a \tilde{n}_a|$ number of independent lumps, each of which should contain a single fundamental monopole. In this sense, there exist only k species of monopoles, just as predicted by the topological argument above. For more details, see Ref. [4].

A dyonic excitation of an $\boldsymbol{\beta}_a^*$ monopole leads to an electric charge proportional to $\boldsymbol{\beta}_a$. Simply as a matter of convenience, we choose to write the long range electric field thereof as follows,

$$\mathbf{E}_a = q_a \frac{\beta_{ai}H_i}{\boldsymbol{\beta}_a^2}\frac{e\,(\mathbf{x}-\mathbf{x}_a)}{4\pi\,|\mathbf{x}-\mathbf{x}_a|^3}, \tag{11}$$

where $\mathbf{x}_a$ is the position of the dyon. With this normalization, the "electric charge" q_a is conjugate to a $U(1)$ angle ξ_a of period $2\pi/\boldsymbol{\beta}_a^2$. This collective coordinate ξ_a together with the position vector $\mathbf{x}_a$ originates from the four zero modes of the given fundamental monopole.

We will not repeat here the derivation of the asymptotic form of the moduli space metric, which can be found in Ref. [10]. It suffices to say that after obtaining the low energy effective Lagrangian, one may trade off the q_a's in favor of their canonical conjugate ξ_a's [1] and reach a purely kinetic form of the Lagrangian in all collective coordinates. From this, one reads off the metric coefficients. For n monopoles of magnetic charges $\{\boldsymbol{\beta}^*_a\}$, the metric is

$$\mathcal{G} = M_{ab} d\mathbf{x}_a \cdot d\mathbf{x}_b + \frac{16\pi^2}{e^4}(M^{-1})_{ab}(d\xi_a + \mathbf{W}_{ac}\cdot d\mathbf{x}_c)(d\xi_b + \mathbf{W}_{bd}\cdot d\mathbf{x}_d), \tag{12}$$

where the $n \times n$ matrix M is

$$\begin{aligned} M_{aa} &= m_a - \sum_{c\neq a} \frac{4\pi\boldsymbol{\beta}^*_a \cdot \boldsymbol{\beta}^*_c}{e^2 r_{ac}}, \\ M_{ab} &= \frac{4\pi\boldsymbol{\beta}^*_a \cdot \boldsymbol{\beta}^*_b}{e^2 r_{ab}} \qquad\qquad \text{if } a \neq b, \end{aligned} \tag{13}$$

with the m_a's being monopole masses and $r_{ab} \equiv |\mathbf{x}_a - \mathbf{x}_b|$. The vector potential $\mathbf{W}_{ab}$ is given by

$$\begin{aligned} \mathbf{W}_{aa} &= -\sum_{c\neq a} \boldsymbol{\beta}^*_a \cdot \boldsymbol{\beta}^*_c \mathbf{w}_{ac}, \\ \mathbf{W}_{ab} &= \boldsymbol{\beta}^*_a \cdot \boldsymbol{\beta}^*_b \mathbf{w}_{ab} \qquad\qquad \text{if } a \neq b, \end{aligned} \tag{14}$$

while $\mathbf{w}_{ac}$ is the abelian vector potential of a negative unit Dirac monopole at $\mathbf{x}_c$, evaluated at $\mathbf{x}_a$. This asymptotic form of the metric does possess the n $U(1)$ symmetry, just as expected; the periodic coordinate ξ_a never appears in the metric coefficients, so the shift $\xi_a \to \xi_a + \mathit{constant}$ is a symmetry of the metric for each a.

Here, it serves a useful purpose to consider the simplest cases with a pair of fundamental monopoles. First suppose the group was $SU(2)$ broken to $U(1)$. There is only one kind of monopole, so consider a pair of identical monopoles. Then $\boldsymbol{\beta}^*_1 = \boldsymbol{\beta}^*_2$, so that their inner product is a positive number. The 2×2 matrix M becomes a finite degenerate one for sufficiently small r_{12}, which leads to a curvature singularity in the metric $\mathcal{G}$. This divergence is of course a clear indication that the asymptotic approximation breaks down for small intermonopole distances. As was emphasized earlier, the symmetry consideration also rules out $\mathcal{G}$ as the exact metric in this case. The two $U(1)$ isometries of $\mathcal{G}$ generated by independent shifts of ξ_1 and ξ_2 translate into two conserved $U(1)$ electric charges, while the exact metric can inherit at most one $U(1)$ symmetry from the unbroken group.

For an interacting pair of distinct monopoles (which is possible for any simple group larger than $SU(2)$), however, the exact metric actually inherits two $U(1)$ symmetries [9, 16]. Furthermore, $\boldsymbol{\beta}_1^* \cdot \boldsymbol{\beta}_2^* < 0$, and this ensures that for any nonzero separation r_{12}, the matrix M is nonsingular and the metric is smooth. The apparent singularity at $r_{12} = 0$ is a coordinate singularity which can be removed by using $\sqrt{r_{12}}$ as the new radial coordinate instead. Thus, there appears no obvious physical reason why this approximate metric $\mathcal{G}$ should receive short distance corrections.

To understand the geometry of this metric for general number of monopoles n, it is useful to separate out a trivial part of the metric. The three overall translations as well as one global phase variable χ, the excitation of which can lead to a BPS saturated dyonic state, generically span a flat four-dimensional Euclidean space R^4, and decouple from the rest. To isolate the interacting part of the metric $\mathcal{G}$, let us assume without loss of generality that the distinct simple roots $\{\boldsymbol{\beta}_a\}$ span a connected subdiagram of the Dynkin diagram of G. Let the $n-1$ links between the adjacent pairs of simple roots be labeled by $A, B, \ldots$, then with an appropriate coordinate redefinition such as

$$\mathbf{r}_A = \mathbf{x}_a - \mathbf{x}_b, \qquad \boldsymbol{\beta}_a \text{ and } \boldsymbol{\beta}_b \text{ connected by the link } A, \tag{15}$$

one finds the following nontrivial part of the metric that describes relative motion of the n monopoles with respect to its center-of-mass [10],

$$\begin{aligned} \mathcal{G}_{\rm rel} &= C_{AB}\, d\mathbf{r}_A \cdot d\mathbf{r}_B \\ &+ \frac{4\pi^2 \lambda_A \lambda_B}{e^4} (C^{-1})_{AB} \,(d\psi_A + \mathbf{w}(\mathbf{r}_A)\cdot d\mathbf{r}_A)(d\psi_B + \mathbf{w}(\mathbf{r}_B)\cdot d\mathbf{r}_B). \end{aligned} \tag{16}$$

The $(n-1)\times(n-1)$ matrix C is

$$C_{AB} = \mu_{AB} + \delta_{AB}\, \frac{2\pi\lambda_A}{e^2 r_A}, \tag{17}$$

where μ_{AB} can be interpreted as a reduced mass matrix and $\lambda_A \equiv -2\boldsymbol{\beta}_a^* \cdot \boldsymbol{\beta}_b^* > 0$ encodes the strength of interaction between the pair connected by the A-th link. Finally, $\mathbf{w}(\mathbf{r}_A) = \mathbf{w}_{ab}$, and ψ_A is a $U(1)$ phase angle of period 4π for all A.[3]

For all n, this metric $\mathcal{G}_{\rm rel}$ is again smooth everywhere and admits exactly the right amount of symmetries. Furthermore, the origin, $r_A = 0$ for all A, is a very special point that deserves a

[3] The total moduli space is not a simple product of the relative part and the "center-of-mass" part R^4. Instead, one must mod out by an integer group $\mathbf{Z}$ that acts on χ and the ψ_A's as a translation [9].

further consideration. It is not only invariant under the spatial rotation but also under the $n-1$ $U(1)$ phase shifts $\psi_A \rightarrow \psi_A + constant$. But this is precisely what one should expect on the exact moduli space: The sum $\sum \boldsymbol{\beta}^*_a$ is equal to $\boldsymbol{\alpha}^*$ for some positive root $\boldsymbol{\alpha}$, so there exists an $SU(2)$ embedded, spherically symmetric, composite monopole solution [4] that is also invariant under all $U(1)$ generators orthogonal $\alpha_i H_i$, which implies an maximally symmetric point on the relative part of the exact moduli space, namely the origin. This is the third compelling evidence that $\mathcal{G}$ ($\mathcal{G}_{\rm rel}$) is in fact the exact moduli space metric for any set of distinct fundamental monopoles [10].

This conjecture has received independent supports lately (for unitary gauge groups). M. Murray [13] demonstrated that the multi-monopole metric derived from the Nahm data [14] (which is believed to be an isometric mapping of BPS monopole configurations) coincides with $\mathcal{G}$. In a more recent work, G. Chalmers [15] exploited the n $U(1)$ gauge isometries to argue that $\mathcal{G}$ is indeed the only smooth hyper-Kähler metric that possesses the right symmetry properties as well as the appropriate asymptotic structure.

In particular, when $n=2$ so that the relative moduli space is four-dimensional, the method developed by Atiyah and Hitchin carries over almost verbatim, regardless of the gauge group. The symmetry of the BPS equation dictates that the metric be hyper-Kähler whose three complex structures rotate under the spatial rotation of the two monopoles, with the latter generating an isometry of the moduli space that has three dimensional orbits generically. The only nontrivial possibilities that fit this criteria are the Atiyah-Hitchin manifold with the symmetry group $SO(3)$ and the Euclidean Taub-NUT manifold with $SU(2)\times U(1)$. Only the Taub-NUT is consistent with the expected amount of symmetry [9, 16, 17] and also its metric $\mathcal{G}_{TN}$ has the right sign to asymptote to the $\mathcal{G}_{\rm rel}$. Now the point is, the approximate form $\mathcal{G}_{\rm rel}$ is actually identical to this exact moduli space metric $\mathcal{G}_{TN}$.

Once we have the exact moduli space, the remaining task boils down to solving a supersymmetric quantum mechanics on that manifold. Furthermore, only $\mathcal{G}_{\rm rel}$ enters the discussion of purely magnetic bound states. To quantize the dynamics, we follow Witten [18]: Because of the supersymmetric nature of the monopoles, each bosonic collective coordinate $z^\mu \in \{\mathbf{r}_A, \psi_A\}$ is accompanied by a single complex fermionic coordinate η^μ and its complex conjugate $\tilde{\eta}^\mu$ [19]. The grassman

algebra obeyed by η^μ is simply

$$\begin{aligned}
\{\eta^\mu, \eta^\nu\} &= 0, \\
\{\tilde{\eta}^\mu, \tilde{\eta}^\nu\} &= 0, \\
\{\tilde{\eta}^\mu, \eta^\nu\} &= \mathcal{G}^{\mu\nu}_{\rm rel}.
\end{aligned} \tag{18}$$

For each μ, $\tilde{\eta}^\mu$ is an creation operator and η^μ an annihilation operator. There are $4(n-1)$ such pairs. The Hilbert space $\mathcal{H}$ is then decomposed to $\oplus_{p=0}^{4(n-1)}\mathcal{H}_p$ where p is the fermion number. The complex supersymmetry generator

$$Q = \tilde{\eta}^\mu \nabla_\mu \tag{19}$$

maps $\mathcal{H}_p$ to $\mathcal{H}_{p+1}$ while its complex conjugate

$$\tilde{Q} = -\eta^\mu \nabla_\mu \tag{20}$$

maps $\mathcal{H}_p$ to $\mathcal{H}_{p-1}$. The similarity with the de Rham complex is in fact exact, and this Hilbert space has one-to-one correspondence with the space of forms on the moduli space where Q is identified with the exterior derivative d and $\tilde{Q}$ with its adjoint $d^\dagger$. The analogy is complete with the observation that the Hamiltonian is simply a square of these generators

$$H = Q\tilde{Q} + \tilde{Q}Q = dd^\dagger + d^\dagger d. \tag{21}$$

In nonrelativistic quantum mechanics, the eigenvalues of the Hamiltonian is the total energy minus the rest mass, so a threshold bound state must be annihilated by H. Thus, a threshold bound state is nothing but a square integrable harmonic form on the relative part of the moduli space.

For $n = 2$, i.e., on the Taub-NUT manifold, a unique normalizable Harmonic form has been found [9, 16]. The Hamiltonian H can be rewritten as

$$H = \nabla^\mu \nabla_\mu + \mathcal{R} \tag{22}$$

with an appropriate curvature term $\mathcal{R}$. On the Taub-NUT manifold, the curvature piece is trivial on 0-forms, 1-forms, and self-dual 2-forms. Then, a vanishing theorem can be formulated despite the noncompact nature of the manifold, which applies to all sectors of the Hilbert space except that of anti-self-dual 2-forms [9]. A normalizable harmonic form, if any, has to be an anti-self-dual 2-form. It is then a matter of solving first order differential equations to find the unique normalizable

harmonic form, or equivalently the threshold bound state. For $G = SU(3)$, this would be dual to the third, most massive vector meson.

There is a very suggestive way of writing this harmonic 2-form Ω_2:

$$\Omega_2 = dK_1, \quad \text{with } K_1 = \frac{\partial}{\partial \psi_1}, \tag{23}$$

where the last equality is via the isomorphism induced by the metric. On a Ricci-flat manifold, an exterior derivative of any Killing one-form is always harmonic because the divergence of it is by virtue of the Killing equations proportional to the Ricci tensor. On the other hand, a hyper-Kähler metric is automatically a Calabi-Yau, so that the moduli space is Ricci-flat. The only remaining question is the normalizability which should be checked on individual basis.

An obvious generalization to $n > 2$, is then to consider the following $2(n-1)$-form,

$$\Omega_{2(n-1)} = dK_1 \wedge dK_2 \wedge \cdots \wedge dK_{n-1}, \tag{24}$$

where K_A is again the 1-form obtained from the Killing vector field $\partial/\partial\psi_A$. This middle form is obviously closed, and its normalizability can be shown easily. Is it co-closed as well? According to Gibbons [20], this middle form is in fact (anti-)self-dual, in which case the closedness automatically implies the co-closedness.

For the gauge groups $SU(k+1)$, the above construction with $n \leq k$ reproduces all of the missing $(\dim G - k)/2 - k = (k^2 - k)/2$ magnetic states which, together with the k fundamental monopoles, are dual to the massive vector mesons. For other gauge groups, it does not reproduce all of the missing states; although a substantial part of them are recovered this way, others involve two or more identical monopoles along with distinct ones. The explicit form of the moduli space metric for the latter cases are unknown except in asymptotic regions.

An interesting generalization concerns monopoles with nonabelian magnetic charges. Such monopoles appear naturally when the unbroken gauge group is not $U(1)^k$ but rather contains a nonabelian factor. In the presence of long range nonabelian magnetic fields, the quantum mechanics of monopoles are often quite subtle because of some nonnormalizable zero modes [21]. Nevertheless, for a collection of monopoles whose total nonabelian magnetic charge vanishes [22], one may proceed to find the moduli space. It turns out that the moduli space in such a case is simply an appropriate massless limit of its counterpart in the maximally broken case [23].

The simplest case where one expects a threshold bound state of such monopoles, arises from a partially broken unitary group. Suppose $G = SU(k+1)$ gauge group is broken to $H = U(1) \times SU(k-1) \times U(1)$. The vacuum manifold G/H has the second homotopy group of $\mathbf{Z}^2$, so there are two species of massive fundamental monopoles, each of which is charged with respect to one of the $U(1)$'s and also carries an $SU(k-1)$ magnetic flux. There is a $4k$-parameter family [24] of solutions that contain one of each topological soliton, and where the total nonabelian magnetic flux vanishes at infinity.

Another way of looking at such solutions is to regard them as a massless limit of the configurations with the magnetic charge $\sum_{a=1}^{k} \boldsymbol{\beta}_a^*$ which would be k-monopole solutions in the maximally broken case. By taking the limit where all monopoles except $\boldsymbol{\beta}_1^*$ and $\boldsymbol{\beta}_k^*$ become massless, we obtain the desired configuration. Then we may start with the metric $\mathcal{G}$ for $n = k$ and $G = SU(k+1)$, allow $m_a \to 0$ for $a = 2, \ldots, k-1$, and end up with the right moduli space of dimension $4k$. The $4(k-1)$-dimensional hyper-Kähler metric on the relative part of this moduli space is simply a degenerate version of $\mathcal{G}_{\rm rel}$ with all entries of the reduce mass matrix, μ_{AB}, equal to one another [23].[4]

This alternate viewpoint also suggests that each topological soliton should be in one of the two defining representations of the magnetic $SU(k-1)$ group. The duality relates them to the two families of degenerate $k-1$ massive vector mesons, each in one of the two defining representations of the unbroken electric $SU(k-1)$. In addition, there exists a color-singlet vector meson. This is the most massive of all vector mesons, so its magnetic counterpart must be again realized as a threshold bound state of the two massive fundamental monopoles. The corresponding harmonic form on the $4(k-1)$-dimensional relative moduli space is yet to be found.

In summary, a very plausible candidate for the exact moduli space metric is found for all collections of distinct fundamental monopoles. For any pair of distinct monopoles, the conjecture can be easily confirmed using the method of Atiyah and Hitchin. Also the resulting multi-monopole moduli space is reconsidered in two recent papers, where proofs of the conjecture are offered for unitary gauge groups. The apparent conflict between the duality hypothesis and the magnetic soliton spectrum is partially resolved, as a substantial fraction of the missing magnetic states are

[4] Such a metric is also known as the Taubian-Calabi metric [25].

recovered in the form of quantum threshold bound states of distinct fundamental monopoles. For unitary gauge groups, this actually resolves the conflict completely. A generalization to the cases with unbroken nonabelian gauge groups is also initiated but more remain to be studied.

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